Helical waves on a vortex filament

Valery P. Dmitriyev

Lomonosov University, P. O. Box 160, Moscow 117574, Russia

A small amplitude helical wave spreading along an unstretchable vortex filament in a perfect fluid obeys the linear Schrödinger equation. Taking into account elastic properties of the filament leads to the Klein-Gordon equation.

Introduction

I suggest an ideal mechanical system whose description is similar to equations that qovern the motion of a quantum particle. This system is a vortex filament in a perfect fluid.

A vortex filament

We consider a vortex tube in an ideal fluid. It can be imagined in the following way. Let a viscous fluid be pierced by a pin that spins about its axis, causing a circular motion in the fluid. The motion persists if we withdraw the pin and the fluid viscosity is removed. A very thin vortex tube will be referred to as a vortex filament.

Following the terminology customary in hydrodynamics we say that a vortex induces a velocity field in the fluid. A vortex moves with the flow of the fluid so that two straight vortex filaments rotate around each other. If we approximate small segments of a curve by straight lines, we conclude that two portions of a vortex filament will affect each other in the same way. The self-induction of a curved vortex filament causes it to evolve.

If we assume that only adjacent parts of the filament influence each other, the following law of motion of a bent vortex filament can be deduced (see, for example, [1]). The velocity $\mathbf{u}$ of the filament is proportional to the filament's curvature κ and is directed along the binormal vector:

$$\mathbf{u} = \nu\kappa\,\mathbf{e}\times\mathbf{n}\,, \tag{1}$$

where ν is a constant, the coefficient of local self-induction, that characterizes a given vortex filament, $\mathbf{n}$ is a principal normal and the tangent vector $\mathbf{e}$ is assumed to be parallel to the filament's vorticity (see Fig.1).

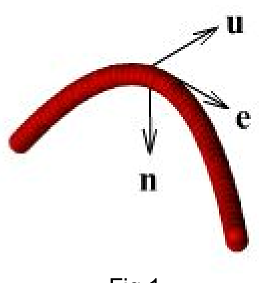

Fig.1

The drift $\mathbf{u} = \partial\mathbf{r}/\partial t$ of a bent vortex filament. Other vectors show the direction $\mathbf{e} = \partial\mathbf{r}/\partial l$ of the filament's vorticity and its curvature $\kappa\mathbf{n} = \partial^2\mathbf{r}/\partial l^2$.

In order to express Eq. (1) in Cartesian coordinates we define the spatial curve in terms of the radius vector $\mathbf{r}$ as a function of the parameter l; for a moving curve, there is a further dependence $\mathbf{r}(l,t)$ on time t:

$$\mathbf{r} = x(l,t)\mathbf{i}_x + y(l,t)\mathbf{i}_y + z(l,t)\mathbf{i}_z\,. \tag{2}$$

Taking for l the length measured along the curve from some fiducial point we may define the unit tangent vector

$$\mathbf{e} = \frac{\partial \mathbf{r}}{\partial l} \ , \tag{3}$$

and the curvature

$$\kappa \mathbf{n} = \frac{\partial \mathbf{e}}{\partial l} \ . \tag{4}$$

The torsion τ of the curve is defined through the binormal $\mathbf{e} \times \mathbf{n}$ by

$$\tau \mathbf{n} = -\frac{\partial (\mathbf{e} \times \mathbf{n})}{\partial l} \ . \tag{5}$$

(see, for example, [2]). The velocity $\mathbf{u}$ of the liquid element of the filament is given by

$$\mathbf{u} = \frac{\partial \mathbf{r}}{\partial t} . \tag{6}$$

We are interested in the case for which the filament deviates only slightly from the rectilinear configuration. Let the filament be directed along the x axis. The smallness of the deviation from the x axis implies that

$$\left|\frac{\partial y}{\partial x}\right| \ll 1, \quad \left|\frac{\partial z}{\partial x}\right| \ll 1 . \tag{7}$$

On account of (7), the corresponding terms will be neglected in further considerations. Then we have for the arc's length

$$dl = \left[1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2\right]^{1/2} dx \approx dx , \tag{8}$$

and (2) is reduced to

$$\mathbf{r} = x\mathbf{i}_x + y(x,t)\mathbf{i}_y + z(x,t)\mathbf{i}_z . \tag{9}$$

Using (9) and (8) in (3), (4) and (6) gives

$$\mathbf{e} = \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i}_x + \frac{\partial y}{\partial x}\mathbf{i}_y + \frac{\partial z}{\partial x}\mathbf{i}_z \approx \mathbf{i}_x , \tag{10}$$

$$\kappa \mathbf{n} = \frac{\partial \mathbf{e}}{\partial x} = \frac{\partial^2 y}{\partial x^2}\mathbf{i}_y + \frac{\partial^2 z}{\partial x^2}\mathbf{i}_z , \tag{11}$$

$$\mathbf{u} = \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial y}{\partial t}\mathbf{i}_y + \frac{\partial z}{\partial t}\mathbf{i}_z . \tag{12}$$

We see from (11) and (12) that the normal (4) and velocity (6) lie in the yz plane. Substituting (10)-(12) into Eq. (1) we get

$$\frac{\partial y}{\partial t}\mathbf{i}_y + \frac{\partial z}{\partial t}\mathbf{i}_z = \nu \mathbf{i}_x \times \left(\frac{\partial^2 y}{\partial x^2}\mathbf{i}_y + \frac{\partial^2 z}{\partial x^2}\mathbf{i}_z\right). \tag{13}$$

According to (13), the direction of the velocity $\mathbf{u}$ can be found rotating $\mathbf{n}$ around the x axis by 90^o from y to z axis. This vector algebra enables us to express Eq. (13) in a complex-valued form.

We define the complex amplitude

$$\Phi = y + iz . \tag{14}$$

Equation (13) is isomorphic to

$$\frac{\partial \Phi}{\partial t} = i\nu \frac{\partial^2 \Phi}{\partial x^2} , \tag{15}$$

where the imaginary unit i corresponds to the vector operator $\mathbf{i}_x \times$. The Frenet-Serret formulae (4) and (5) look like

$$\kappa n = \frac{\partial^2 \Phi}{\partial x^2} \tag{16}$$

and

$$\tau n = -i\frac{\partial n}{\partial x} \tag{17}$$

in the complex-valued representation (14) with the complex value n standing for the vector $\mathbf{n}$. In view of (16), (17) the solution to Eq. (15) is given by

$$\Phi = a\exp\left[i\left(\tau x - \omega t\right)\right] \tag{18}$$

with

$$\omega = \nu\tau^2 \tag{19}$$

and

$$\kappa = a\tau^2, \tag{20}$$

the principal normal being

$$n = -\exp\left[i\left(\tau x - \omega t\right)\right]. \tag{21}$$

Using (14) and (18) in (7) we find that the Schrödinger equation (15) is valid provided that

$$a\tau << 1, \tag{22}$$

i.e. when the curvature is much less than the torsion:

$$\kappa << \tau . \tag{23}$$

The curve with constant curvature and torsion is referred to as a helix (see Fig.2 where the case of $\kappa > \tau$ is shown for better visulization). The helical configuration of a vortex filament rotates counter-clockwise around the x axis looking in the direction of the x axis with the angular velocity (19). A curvilinear configuration of an unstretchable filament can be obtained from the straight filament by adding a filament's segment to it. For a helix it amounts to $\pi a\kappa/\tau$ per a turn of the spiral.

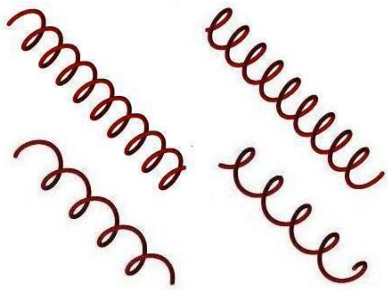

Fig.2
Left-hand screw helices (left) and right-hand screw helices (right);
$\kappa = 4\tau$ (top) and $\kappa = 2\tau$ (bottom).

We have found that for small deviations from the straight line the amplitude (14) of a vortex filament obeys the linear Schrödinger equation. Next, we may unify the description rendering it into a positionally invariant form (and thereby simplify its derivation). To this end we will differentiate Eq. (1) twice with respect to x and use in the result Eqs. (6), (3) and (4) with (8) assuming that $\mathbf{e} = \mathbf{i}_x$. This gives the form of the motion law (1) needed

$$\frac{\partial(\kappa\mathbf{n})}{\partial t} = \nu\,\mathbf{i}_x \times \frac{\partial^2(\kappa\mathbf{n})}{\partial x^2}. \tag{24}$$

Rewriting Eq.(24) in complex values by the above receipt we immediately get the linear Schrödinger equation

$$\frac{\partial\Psi}{\partial t} = i\nu\frac{\partial^2\Psi}{\partial x^2}. \tag{25}$$

for the complex-valued counterpart

$$\Psi = \kappa n \tag{26}$$

of the curvature vector $\kappa\mathbf{n}$. Substituting (21) into (26) we find the helical form complied with Eq.(25):

$$\Psi = -\kappa \exp\left[i\left(\tau x - \omega t\right)\right]. \tag{27}$$

Note that we did not presuppose constancy of κ in the derivation of (24).

The amplitude invariant form (27) corresponds to the asymptotics, given by the condition (7), of the general solution of Eq.(1) describing a kink on the vortex filament [3]. For relatively large torsion, the nonlinear solution has the form of a spiral bulge on the filament (Fig. 3). This configuration can be loosely designated as a wave packet because of its topological similarity to the finite helical configuration obtained by the linear superposition of plain waves (simple helices) with a dispersion of wave number τ. For the sake of visualization, the case $\kappa > \tau$ is shown in Fig. 3.

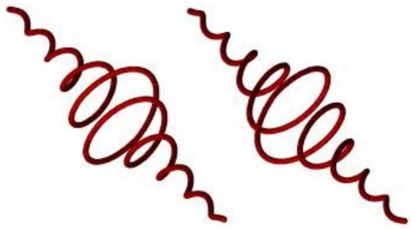

Fig.3

The wave packets comprised of left-handed (left) and right-handed (right) helices, $\kappa = 10\tau$.

Substituting (18) into the right-hand part of Eq.(15) and using in it (27) with (20) we find

$$\frac{\partial \Phi}{\partial t} = i\nu\Psi . \tag{29}$$

By virtue of (29) we may somehow interpret the wave function Ψ in physical terms as the velocity that the helix rotates about the screw axis (see for details [4, 5]).

An elastic vortex filament

The elastic stretching of the vortex filament can be also taken into account. To describe it, we will first consider the motion of an elastic string.

The behavior of a rectilinear string that can be stretched elastically causing a displacement $\mathbf{s}$ in the direction of y and z axis is known to obey the d'Alembert equation

$$\frac{\partial^2 \mathbf{s}}{\partial t^2} = c^2 \frac{\partial^2 \mathbf{s}}{\partial x^2}. \tag{29}$$

A solution to Eq. (29) is given by two independent functions:

$$\mathbf{s} = y(x-ct)\mathbf{i}_y + z(x-ct)\mathbf{i}_z . \tag{30}$$

Equation (30) specifies a hump that moves along the x axis with the constant velocity c.

In order to describe the object that combines in itself properties of a vortex filament and of an elastic string we unite Eqs. (15) and (29) into a single equation. To this end let us rewrite Eq. (29) in a complex-valued form:

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \frac{\partial^2 \Phi}{\partial x^2}, \tag{31}$$

where Φ is given by Eq. (14).

We next take advantage of the fact that the rotating helix (18) also provides a particular solution to Eq. (31). The left-hand part of Eq. (29) or (31) has the meaning of the acceleration. In order to find an addition to the acceleration due to self-induction of a vortex filament we take the second derivative of (18) with respect to time. That gives

$$\frac{\partial^2 \Phi}{\partial t^2} = -\omega^2 \Phi . \tag{32}$$

Adding the right-hand part of Eq. (32) to the right-hand part of Eq. (31) and renaming the coefficient before Φ to ω_0^2 we get the equation

$$\frac{\partial^2\Phi}{\partial t^2} = c^2\frac{\partial^2\Phi}{\partial x^2} - \omega_0^2\Phi \tag{33}$$

which describes both the elastic stretching and self-induction of a vortex filament.

The form (18) describing a rotating helix satisfies the Klein-Gordon equation (33) with the following "relativistic" relation for the angular velocity ω of rotation

$$\omega^2 = c^2\tau^2 + \omega_0^2 . \tag{34}$$

In the right-hand part of Eq. (34) the term $c\tau$ corresponds to the angular velocity of rotation of an elastic string and $\omega_0 = \nu\tau_0^2$ to that of the unstretched vortex filament.

Conclusion

Vortex filaments are structural constituents of turbulence in a fluid [6]. Perturbation waves on vortices represent a secondary formation known as soliton turbulence. A kink on a straight vortex filament evolves according to the nonlinear Schrödinger equation [3]. Schrödinger dynamics was shown to be valid not only for an isolated vortex filament, but also for perturbation waves on vortex cores in general [7]. So, the nonlinear Schrödinger equation can be used as a model of soliton turbulence [8, 9]. In the present discourse only the case of fully thermalized soliton was considered.

We may thus conclude that the vortical turbulence provides a substratum for the deterministic Schrödinger and Klein-Gordon fields.

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